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ELASTIC–PLASTIC STATE OF A WEDGE WITH LIMITING RESISTANCE TO SHEAR AND SEPARATION

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The equilibrium of an acute-angled wedge is examined under plane strain conditions under the action of a uniformly distributed load q applied along the normal to one of its faces (Fig. 1) in the presence of strength limits $k > 0$, $d > 0$ to shear and separation [1] such that the tangential and normal stress components would satisfy the conditions $\tau_{max} \leqslant k$, $\sigma_{max} \leqslant d$. The domain in which the maximal stress components do not reach these limits will be considered elastic.

In contrast to an elastic–plastic wedge (with a limiting resistance to just shear [2]), in this case the size of the limit state zone depends on not only the magnitude but also the direction of the load ($q > 0$, $q < 0$). Intervals of the load q that correspond to different qualitative states of the wedge are determined.

1. In the general case the wedge is separated into three zones I-III (Fig. 1) in which shear, elastic, and separation states, respectively, occur. The equations for the stress in zone I are of hyperbolic type and have two orthogonal families of rectilinear characteristics inclined to the free boundary at the angle $\pi/4$, while the equations in zone III are of parabolic type and have one family of characteristics orthogonal to the principal stress [1]. A uniform stressed state is realized in zones I and III and the boundaries with zone II are rectilinear.

Let us ascribe the superscripts minus and plus, respectively, to the stress tensor components in zones I and III. Here the principal stresses have the definite values

$$\sigma_1^- = 0, \quad \sigma_2^- = -2k, \quad \sigma_1^+ = d, \quad \sigma_2^+ = -q.$$

The stress states in zones I and III are interpreted by Mohr diagrams in Fig. 2. The normal and tangential stress tensor components can evidently be defined in terms of the principal stresses on the lines OB and OC separating the three zones and making the angles α and β with the wedge faces (see Fig. 1). We have in the r, θ polar coordinates

$$\sigma_\theta^-, \sigma_r^- = k(\pm\cos 2\alpha - 1), \quad \tau_{r\theta}^- = k\sin 2\alpha, \quad \sigma_\theta^+, \sigma_r^+ = p \mp \rho\cos 2\beta, \quad \tau_{r\theta}^+ = \rho\sin 2\beta, \tag{1.1}$$

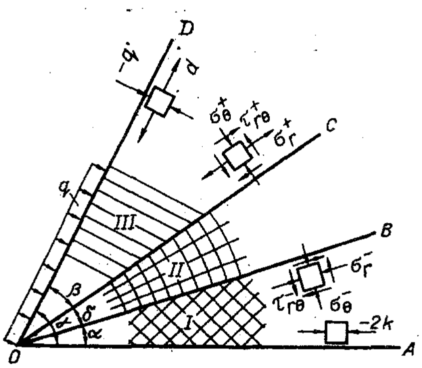

Fig. 1

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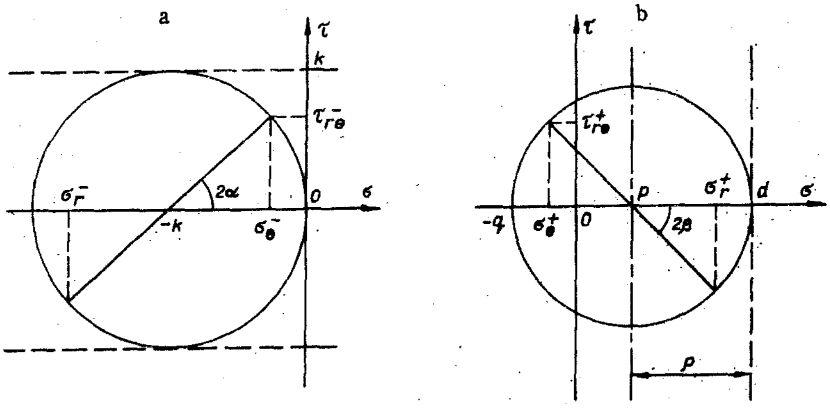

Fig. 2

where $p = (1/2)(d - q)$, $\rho = (1/2)(d + q)$ are coordinates of the center and radius of the Mohr circle in Fig. 2b.

Following [3], we represent the stress components in the elastic zone II in the form

$$\sigma_r, \sigma_\theta = A - 2D\theta \pm (B \sin 2\theta + C \cos 2\theta), \ \tau_{r\theta} = D + B \cos 2\theta - C \sin 2\theta. \tag{1.2}$$

We agree to measure the angle θ so that $\theta = 0$ on the boundary OB and $\theta = \gamma - \alpha - \beta = \delta$ on OC. Then from the condition of stress continuity on the lines $\theta = 0$ and $\theta = \delta$ a system of six equations follows from (1.1) and (1.2)

$$p \pm \rho \cos 2\beta = A - 2D\delta \pm (B \sin 2\delta + C \cos 2\delta),$$
$$k(\mp \cos 2\alpha - 1) = A \pm C, \ k \sin 2\alpha = D + B,$$
$$\rho \sin 2\beta = D + B \cos 2\delta - C \sin 2\delta.$$

We use four of them to eliminate the arbitrary constants

$$A = -k, \ C = -k \cos 2\alpha, \ D = -(k + p)/2\delta,$$
$$B = (\rho \cos 2\beta + k \cos 2\alpha \cos 2\delta)/\sin 2\delta.$$

The remaining two equations reduce to the form

$$\begin{aligned} k \cos 2(\gamma - \beta) + \rho \cos 2\beta &= g(2\delta)^{-1} \sin 2\delta, \\ \rho \cos 2(\gamma - \alpha) + k \cos 2\alpha &= g(2\delta)^{-1} \sin 2\delta, \end{aligned} \tag{1.3}$$

where $g = k + p$.

Let us introduce the notation $\chi = \beta - \alpha$. Subtracting (1.3) term by term, we find

$$\chi = \operatorname{arctg}\left(\frac{k-\rho}{k+\rho} \operatorname{tg} \gamma\right). \tag{1.4}$$

The angular dimensions of the separation and shear zones are determined from the geometric condition $\gamma = \alpha + \beta + \delta$ with (1.4) taken into account:

$$\alpha = (1/2)(\gamma - \delta - \chi), \ \beta = (1/2)(\gamma - \delta + \chi).$$

For the final solution of the problem under consideration about the equilibrium of an elastic–plastic wedge for a limiting resistance to shear and separation there remains to express the function $\delta(q)$ whose specific form depends on the interval in which the external load q is found, from (1.3).

2. A load interval $0 < q < q_1$ evidently exists for which the wedge is completely in the elastic state. Here $\alpha = \beta = 0$, $\delta = \gamma$, the stress on the loaded face $\sigma_r = l$ should not reach the limit of the resistance to separation, and on the free face $\sigma_r = -2\varkappa$ should not reach the limit of the resistance to shear, i.e., $0 < l < d$, $0 < \varkappa < k$. Moreover, $\rho = (1/2)(l + q)$, $p = (1/2)(l - q)$. Taking these conditions into account, we have from (1.3)

$$\rho = \varkappa = (1/2)\, q\, (1 - \gamma \operatorname{ctg} \gamma)^{-1}, \ l = q(\gamma^{-1} \operatorname{tg} \gamma - 1).^{-1}$$

If the parameters ρ and $\varkappa$ reach the limit k as the load q grows, then the plasticity condition occurs simultaneously on both faces. Under higher loads, two plastic zones are formed in this case, which abut on the wedge faces and have identical size. Such a limit state is investigated in detail in [2].

However, if the stress l reaches the limit d earlier than ρ reaches the value k, then the separation condition

$$q = q_1 = d(\gamma^{-1}\,\mathrm{tg}\,\gamma - 1),\ \rho = (1/2)(d + q_1) < k$$

occurs on the loaded face of the wedge. This latter condition is expressed in the form of a constraint for the separation strength limit at which the wedge separation strength state under consideration will occur:

$$d < 2k\gamma\,\mathrm{ctg}\,\gamma.$$

It is hence seen that $d \leqslant 0$ for $\gamma \geqslant \pi/2$, i.e., the constant d loses the meaning of the separation strength limit for an obtuse-angled wedge. The zone of the limit separation strength only occurs in an acute-angled wedge.

A limiting separation strength zone abutting on the loaded face *OD* occurs in the load interval $q_1 < q < q_2$, and an elastic state in the rest. The stress on the free face *OA* $\sigma_r = -2\varkappa$ will not reach the limit corresponding to the plasticity state $\sigma_r = -2k$. Consequently, by substituting the parameter $\varkappa$ instead of k, and $\alpha = 0$, $\beta = \gamma - \delta$, we find from (1.3)

$$\varkappa = \rho(\mathrm{ctg}\,\delta\,\sin 2\gamma - \cos 2\gamma),$$
$$\rho = d[1 + \cos 2\gamma + (\delta\,\sin^{-2}\delta - \mathrm{ctg}\,\delta)\sin 2\gamma]^{-1}. \quad (2.1)$$

Since $q = 2\rho - d$, this latter equations expresses the dependence between the dimension δ of the elastic zone and the load q in the interval under consideration. The plasticity condition $\varkappa = k$ occurs for a load $q = q_2$ on the free face. Here $\alpha = 0$, $\beta = \gamma - \delta$ and after elimination of δ, Eqs. (1.3) reduce to the following

$$\frac{\rho \sin 2\gamma}{k - \rho\cos 2\gamma} = \mathrm{tg}\,\frac{(k + d - \rho)\,\rho\sin 2\gamma}{k^2 + \rho^2 + 2k\rho\cos 2\gamma}.$$

The value of ρ satisfying this latter equation permits finding the load $q_2 = 2\rho - d$. For a load in the interval $q_2 < q < q_3$ a limiting shear strength zone abutting on the free face of the wedge and a limiting separation strength zone abutting on the loaded face occur (see Fig. 1). They are separated by an elastic zone which diminishes as q increases and close up at the definite load $q = q_3$. Zones I and III adjoin, i.e., $\delta = 0$, $\alpha + \beta = \gamma$ and we obtain the following condition from (1.3)

$$k\cos 2\alpha + \rho\cos 2\beta = k + p. \quad (2.2)$$

Starting from (1.2), differences in the corresponding stress components can be compiled for the general case $\delta \neq 0$

$$[\tau_{r\theta}] = (k\cos 2\alpha - \rho\cos 2\beta)\,\mathrm{tg}\,\delta,$$
$$[\sigma_r],\ [\sigma_\theta] = (k + p) \pm (k\cos 2\alpha + p\cos 2\beta).$$

In the limit case $\delta \to 0$ the tangential component of the stress remains continuous, since the difference $[\tau_{r\theta}]$ tends to zero. According to (2.2), the normal component σ_θ retains its continuity $[\sigma_\theta] = 0$ but the stress σ_r undergoes a discontinuity $[\sigma_r] = 2(k + p)$.

Therefore, in the limit case of the load $q = q_3$ the elastic zone II degenerates into a line of stress discontinuity. This equilibrium state of the wedge is investigated in [4]. In particular, from the continuity condition for the two stress components $[\sigma_\theta] = 0$ and $[\tau_{r\theta}] = 0$ the following value is found

$$q_3 = d(d/2k + \cos^2\gamma)^{-1}\sin^2\gamma.$$

Eliminating the parameters α and β directly from (1.3), we obtain a dependence between the size of the elastic zone and the load in the interval $q_2 < q < q_3$

$$\delta^{-1}\sin\delta = (1 - dk^{-1} - \rho k^{-1})^{-1}\sqrt{1 - \rho^2 k^{-2} + 2\rho k^{-1}\cos 2\gamma}. \quad (2.3)$$

A graphical interpretation of the dependence $\delta(q)$ on the basis of (2.1) and (2.3) is presented in Fig. 3 for different wedges for identical shear and separation strength limits d, k. The transitions of the quantitative stress changes into qualitative changes of state of the wedge limit resistance that occur under the loads q_1, q_2, q_3 correspond to the points A, B, C.

3. The separation, elasticity, and shear zones also occur under a uniformly distributed load applied along the external normal to the face *OA* (tension). The stress tensor components on the lines *OB* and *OC* will be expressed as follows:

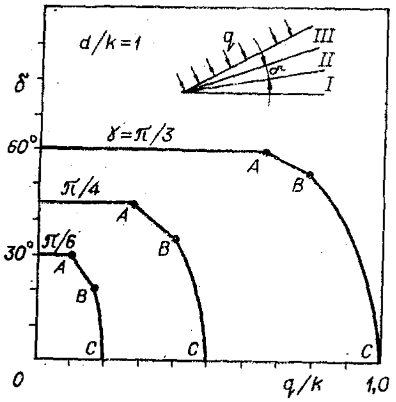

Fig. 3

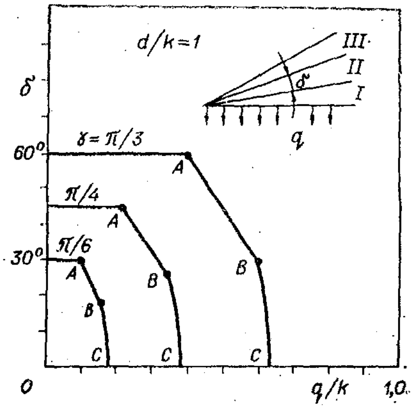

Fig. 4

$$\sigma_\theta^-,\ \sigma_r^- = q - k(1 \mp \cos 2\alpha),\quad \tau_{r\theta}^- = k\sin 2\alpha,$$
$$\sigma_\theta^+,\ \sigma_r^+ = (1/2)\,d\,(1 \pm \cos 2\beta),\quad \tau_{r\theta}^+ = (1/2)\,d\sin 2\beta.$$

The stress components are represented in the form (1.2) in the elastic zone II. As is Sec. 2, we find four constants from the six connection conditions on the lines $\theta = 0$ and $\theta = \delta$:

$$A = q - k,\ C = -k\cos 2\alpha,\ D = -g_1/2\delta,$$
$$B = (\rho_1 \cos 2\beta + k\cos 2\alpha \cos 2\delta)/\sin 2\delta,$$
$$\rho_1 = d/2,\ g_1 = \rho_1 + k - q,$$

and we find two equations (1.3) in which ρ and g should be replaced by ρ_1 and g_1, respectively.

The constants q_1, q_2, q_3, which have the previous meaning (Sec. 2), are found in the form

$$q_1 = d(1 - \gamma \operatorname{ctg}\gamma),$$
$$q_2 = (1/2)\,d + k - \frac{(d^2 + 4k^2 + 4dk\cos 2\gamma)(\gamma - \chi_1)}{2d\sin 2\gamma},$$
$$q_3 = (1/2)d + k - (1/2)\sqrt{d^2 + 4k^2 + 4kd\cos 2\gamma},$$

where χ_1 is determined as χ in (1.4) for $\rho = \rho_1$.

Even here there is evidently three separation, elasticity, and shear zones only for an acute-angled wedge. Indeed, for $\gamma \geqslant \pi/2$, $q_1 \geqslant d$, i.e., the separation condition occurs in an obtuse-angled wedge not only on the *O*D face of zone III but also on the *O*A face of zone I where separation occurs along *O*A. Consequently, for $\gamma \geqslant \pi/2$ the examination of zone I as a shear zone is unacceptable here.

The dependence between the size of the elastic zone δ and the load q in the intervals $q_1 < q < q_2$ and $q_2 < q < q_3$ is given by the respective formulas

$$q = d\left[\sin^2\gamma - \frac{\sin 2\gamma\,(2\delta - \sin 2\delta)}{4\sin^2\delta}\right],$$
$$q = k + \frac{1}{2}d - \frac{\delta\sqrt{d^2 + 4k^2 + 4dk\cos 2\gamma}}{2\sin\delta}.$$

A graphical interpretation of these dependences is represented in Fig. 4. Qualitatively it duplicates the dependences in Fig. 3 but the processes corresponding to points A, B, C set in at lower loads q_1, q_2, q_3 in this latter case. In both cases an insignificant change in the load q as the point C is approached will result in an abrupt change in the elasticity zone δ.

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